

# Cherenkov-like shock waves associated with surpassing the light velocity barrier

G.N. Afanasiev<sup>1</sup> and V.G. Kartavenko

Bogoliubov Laboratory of Theoretical Physics,  
Joint Institute for Nuclear Research,  
Dubna, Moscow District, 141980, Russia

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<sup>1</sup>Corresponding author: Phone: +7 (09621) 63179 Fax: +7 (09621) 65084,  
E-mail: afanasev@thsun1.jinr.ru

## **Abstract**

The effects arising from accelerated and decelerated motion of a point charge inside a medium are studied. The motion is manifestly relativistic and may be produced by a constant uniform electric field. It is shown that in addition to the bremsstrahlung and Cherenkov shock waves, the electromagnetic shock wave arises when the charge particle velocity coincides with the light velocity in the medium. For the accelerated motion this shock wave forming an indivisible entity with the Cherenkov shock wave arrives after the arrival of the bremsstrahlung shock wave. For the decelerated motion the above shock wave detaches from the charge at the moment when its velocity coincides with the light velocity in the medium. This wave existing even after termination of the charge motion of the charge propagates with the light velocity in the medium. It has the same singularity as the Cherenkov shock and is more singular than the bremsstrahlung shock wave. The space-time regions, where these shock waves exist, and conditions under which they can be observed are determined.

# 1 Introduction

Although the Vavilov-Cherenkov effect is a well-established phenomenon widely used in physics and technology (see. e.g., Frank's book [1]), a lot of its aspects remain uninvestigated up to now. In particular, it is not clear how a transition from the sub-light velocity regime to the super-light one takes place. Some time ago (Tyapkin [2], Zrelov et al. [3]) it was suggested that alongside with the usual Cherenkov and bremsstrahlung (*BS*) shock waves, there should exist a shock wave associated with the charged particle passing the medium light velocity  $c_n$ . The consideration presented there was pure qualitative without any formulae and numerical results. It was based on the analogy with the phenomena occurring in acoustics and hydrodynamics. It seems to us that this analogy is not complete and, therefore, it cannot be considered as a final proof.

Usually, treating the Vavilov-Cherenkov effect, one considers the charge motion in an infinite medium with a constant velocity  $v > c_n$ . In the absence of  $\omega$  dispersion, there is no electromagnetic field (*EMF*) before the Cherenkov cone accompanying the charge, an infinite *EMF* on the Cherenkov cone itself and finite values of the *EMF* strengths behind it ([1, 4, 5]). In this case, information concerning the transition effects arising when charge velocity coincides with  $c_n$  is lost (except for the existence of the Cherenkov shock wave (*CSW*) itself).

The accelerated motion of the point charge in a vacuum was first considered by Schott ([6]). Yet, his qualitative consideration was pure geometric, not allowing numerical investigations.

Tamm ([7]) approximately solved the following problem: A point charge rests at a fixed point of medium up to some moment  $t = -t_0$  after which it exhibits the instant infinite acceleration and moves uniformly with the velocity greater than the light velocity in the medium. At the moment  $t = t_0$  the charge decelerates instantly and rests afterwards. Later, this problem was numerically investigated by Ruzicka and Zrelov ([8,9]). The analytic solution of this problem in the absence of dispersion has been found in [5]. However, in all these studies the information concerning the transition effects arising when charge velocity surpasses the light velocity in medium was lost (due to the

instant charge acceleration).

Another possibility is to use smooth accelerated and decelerated charge motion as a tool for the studying the above-mentioned transition effects. Previously, the straight-line motion of a point charge with constant acceleration ( $z = at^2$ ) has been considered in [10]. This motion law is obtained from the relativistic equation

$$m \frac{d}{dt} \frac{v}{\sqrt{1 - \beta^2}} = eE, \quad v = \frac{dz}{dt}, \quad \beta = v/c,$$

( $m$  is the rest mass) for the following electric field directed along the  $z$  axis

$$E_z = \frac{2ma}{e(1 - 4az/c^2)^{3/2}}. \quad (1.1)$$

For the case of accelerated motion it has been found there that two shock waves arise when charge velocity coincides with  $c_n$ . The first of them is the well-known Cherenkov shock wave  $C_M$  having the form of the finite Cherenkov cone and propagating with the velocity of the charge. The second of these waves ( $C_L$ ), closing the Cherenkov cone and propagating with the velocity  $c_n$ , is just the shock wave the existence of which was qualitatively predicted by Tyapkin ([2]) and Zrelov et al.([3]). These two waves form an indivisible entity. As time goes, the dimensions of this complex grow, but its form remains essentially the same. The singularities of the  $C_L$  and  $C_M$  shock waves are the same and much stronger than the singularity of the  $BS$  shock wave arising from the beginning of the charge motion.

For the case of decelerated motion it has been found in the same reference [10] that an additional shock wave arises at the moment when the charge velocity coincides with  $c_n$ . This wave being detached from the charge exists even after termination of its motion. It propagates with the velocity  $c_n$  and has the same singularity as  $CSW$ .

The drawback of this consideration is that the electric field (1.1) maintaining charge motion tends to  $\infty$  as  $z$  approaches  $c^2/4a$ . This singularity makes the creation of electric field (1.1) be rather problematic. This, in turn, complicates the experimental verification of the shock waves mentioned above.

Here, we consider the straight-line motion of a point charge in a constant uniform electric field (which is much easier to create than the singular electric field (1.1)) and evaluate  $EMF$  arising from such a motion. The arising

motion law is manifestly relativistic. We suggest this motion law to be given, disregarding the energy losses and the medium influence on a moving charge.

Qualitatively, we confirm the results obtained in [10] concerning the existence of the shock waves associated with the surpassing the light medium velocity.

In the present approach, we take the refractive index to be independent of  $\omega$ . This permits us to solve the problem under consideration explicitly. The price for disregarding of the  $\omega$  dependence is the divergence of integrals quadratic in the Fourier transforms of field strengths (such as the total energy flux).

This consideration is on the same footing as the first Tamm and Frank papers on the Vavilov-Cherenkov effect in which the dispersion and the influence of energy losses on the uniform point charge motion were not taken into account. In spite of this, these papers correctly predicted the location of the Vavilov-Cherenkov singularity. The subsequent inclusion of dispersion only slightly changed these results.

Another argument for the simplified treatment of the charge accelerated motion (i.e., without  $\omega$  dispersion) is due to Refs. [11] where the uniform motion of a charge in medium with a standard  $\omega$  dependence of electric permittivity ( $\epsilon(\omega) = 1 + \omega_L^2/(\omega_0^2 - \omega^2 + ip\omega)$ ) was considered. It was shown there that such a  $\omega$  dependence of  $\epsilon$  removes singularities of the field strengths and leads to the appearance of many maxima of the radiated energy flux behind the moving charge. However, the main radiation maximum is at the same position as in the absence of  $\omega$  dispersion. Further, despite the  $\omega$  dispersion, the critical charge velocity (independent of frequency and dependent on medium properties) exists below and above of which the radiation spectrum differs drastically. It turns out that for the uniform charge motion the main features of the Cherenkov-Vavilov radiation are the same with and without dispersion. Thus, we hope the same is true for the *EMF* radiated by the accelerated charge moving in medium.

## 2 Statement of the physical problem

Let a point charge move inside the medium with the polarizabilities  $\epsilon$  and  $\mu$  along the given trajectory  $\vec{\xi}(t)$ . Then, its *EMF* at the observation point  $(\rho, z)$  is given by the Lienard-Wiechert potentials (see, e.g., [12])

$$\Phi(\vec{r}, t) = \frac{e}{\epsilon} \sum_i \frac{1}{|R_i|}, \quad \vec{A}(\vec{r}, t) = \frac{e\mu}{c} \sum_i \frac{\vec{v}_i}{|R_i|}, \quad \text{div} \vec{A} + \frac{\epsilon\mu}{c} \dot{\Phi} = 0 \quad (2.1)$$

Here

$$\vec{v}_i = \left( \frac{d\vec{\xi}}{dt} \right)_{t=t_i}, \quad R_i = |\vec{r} - \vec{\xi}(t_i)| - \vec{v}_i(\vec{r} - \vec{\xi}(t_i))/c_n$$

and  $c_n$  is the light velocity inside the medium ( $c_n = c/\sqrt{\epsilon\mu}$ ). The summing in (2.1) is performed over all physical roots of the equation

$$c_n(t - t') = |\vec{r} - \vec{\xi}(t')| \quad (2.2)$$

To preserve the causality, the time of radiation  $t'$  should be smaller than the observation time  $t$ . Obviously,  $t'$  depends on the coordinates  $\vec{r}, t$  of the observation point  $P$ . With the account of (2.2) one gets for  $R_i$

$$R_i = c_n(t - t_i) - \vec{v}_i(\vec{r} - \vec{\xi}(t_i))/c_n \quad (2.3)$$

Consider the motion of the charged point-like particle of the rest mass  $m$  inside the medium according to the motion law ([12])

$$z(t) = \sqrt{z_0^2 + c^2 t^2} + C.$$

It may be realized in a constant electric field  $E$  directed along the  $Z$  axis:  $z_0 = |mc^2/eE| > 0$ . Here  $C$  is an arbitrary constant. We choose it from the condition  $z(t) = 0$ . Therefore,

$$z(t) = \sqrt{z_0^2 + c^2 t^2} - z_0 \quad (2.4)$$

This law of motion, being manifestly relativistic, corresponds to constant proper acceleration [12].

The charge velocity is given by

$$v = \frac{dz}{dt} = c^2 t (z_0^2 + c^2 t^2)^{-1/2}.$$

Clearly, it tends to the light velocity in vacuum as  $t \rightarrow \infty$ . The retarded times  $t'$  satisfy the following equation:

$$c_n(t - t') = [\rho^2 + (z + z_0 - \sqrt{z_0^2 + c^2 t'^2})^2]^{1/2} \quad (2.5)$$

It is convenient to introduce the dimensionless variables

$$\tilde{t} = ct/z_0, \quad \tilde{z} = z/z_0, \quad \tilde{\rho} = \rho/z_0 \quad (2.6)$$

Then,

$$\alpha(\tilde{t} - \tilde{t}') = [\tilde{\rho}^2 + (\tilde{z} + 1 - \sqrt{1 + \tilde{t}'^2})^2]^{1/2}, \quad (2.7)$$

where  $\alpha = c_n/c$  is the ratio of the light velocity in medium to that in vacuum. In order not to overload exposition we drop the tilde signs

$$\alpha(t - t') = [\rho^2 + (z + 1 - \sqrt{1 + t'^2})^2]^{1/2} \quad (2.8)$$

For the treated one-dimensional motion the denominators  $R_i$  are given by

$$R_i = \frac{z_0}{\alpha \sqrt{1 + t_i^2}} [\alpha^2(t - t_i) \sqrt{1 + t_i^2} - t_i(z + 1 - \sqrt{1 + t_i^2})] \quad (2.9)$$

We consider the following two problems :

- I. A charged particle rests at the origin up to a moment  $t' = 0$ . After that it is accelerated in the positive direction of the  $Z$  axis.
- II. A charged particle decelerates moving from  $z = \infty$  to the origin. After the moment  $t' = 0$  it rests there.

It is easy to check that the moving charge acquires the light velocity  $c_n$  at the moments  $t_l = \pm \alpha / \sqrt{1 - \alpha^2}$  for the accelerated and decelerated motion, resp. The position of a charge at those moments is  $z_l = 1 / \sqrt{1 - \alpha^2} - 1$ .

It is our aim to investigate space-time distribution of  $EMF$  arising from such particle motions. For this, we should solve Eq.(2.8). Taking its square we obtain the fourth degree algebraic equation relative to  $t'$ . Solving it, we find space-time domains where the  $EMF$  exists. It is just this way of finding the  $EMF$  which was adopted in [10]. It was shown in the same reference that there is another, much simpler approach for recovering  $EMF$  singularities (it was extensively used by Schott [6]). We seek the zeros of the denominators  $R_i$

entering into the definition of the electromagnetic potentials (2.1). They are obtained from the equation

$$\alpha^2(t - t')\sqrt{1 + t'^2} - t'(z + 1 - \sqrt{1 + t'^2}) = 0 \quad (2.10)$$

We rewrite (2.8) in the form

$$\rho^2 = \alpha^2(t - t')^2 - (z + 1 - \sqrt{1 + t'^2})^2. \quad (2.11)$$

Recovering  $t'$  from (2.10) and substituting it into (2.11) we find the surfaces  $\rho(z, t)$  carrying the singularities of the electromagnetic potentials. They are just shock waves which we seek. It turns out that *BS* shock waves (i.e., moving singularities arising from the beginning or termination of a charge motion) are not described by Eqs. (2.10) and (2.11). The physical reason for this is that on these surfaces *BS* field strengths, not potentials, are singular ([5]). The simplified procedure mentioned above for recovering of moving *EMF* singularities is to find solutions of (2.10) and (2.11) and add to them "by hand" the positions of *BS* shock waves defined by the equation  $r = \alpha t$ ,  $r = \sqrt{\rho^2 + z^2}$ . The equivalence of this approach to the complete solution of (2.8) has been proved in [10] where the complete description of the *EMF* (not only its moving singularities as in the present approach) of a moving charge was given. It was shown there that the electromagnetic potentials exhibited infinite (for the Cherenkov and the shock waves under consideration) jumps when one crosses the above singular surfaces. Correspondingly, field strengths have  $\delta$ -type singularities on these surfaces while the space-time propagation of these surfaces describes the propagation of the radiated energy flux.

### 3 Numerical results

We consider the typical case when the ratio  $\alpha$  of the light velocity in medium to that in vacuum is equal to 0.8.

#### 3.1 Accelerated motion

For the first of the treated problems ( uniform acceleration of the charge resting at the origin up to a moment  $t = 0$ ) only positive retarded times  $t_i$  have a



physical meaning (negative  $t_i$  correspond to the charge resting at the origin). The resulting configuration of the shock waves for the typical observation time  $t = 2$  is shown in Fig.1. We see on it:

- i) The Cherenkov shock wave  $C_M$  having the form of the Cherenkov cone;
- ii) The shock wave  $C_L$  closing the Cherenkov cone and describing the shock wave emitted from the point  $z_l = (1 - \alpha^2)^{-1/2} - 1$  at the moment  $t_l = \alpha(1 - \alpha^2)^{-1/2}$  when the velocity of a charge coincides with the light velocity in medium;
- iii) The *BS* shock wave  $C_0$ .

It turns out that the surface  $C_L$  is approximated with good accuracy by the spherical surface  $\rho^2 + (z - z_l)^2 = (t - t_l)^2$  (shown by the short-dash curve  $C$ ) It should be noted that only the part of  $C$  coinciding with  $C_L$  has physical meaning.

On the internal sides of the surfaces  $C_L$  and  $C_M$  electromagnetic potentials acquire infinite values. On the external side of  $C_M$  lying outside  $C_0$  the magnetic vector potential is zero (as there are no solutions of Eqs. (2.10),(2.11) there), while the electric scalar potential coincides with that of the resting charge. On the external sides of  $C_L$  and on the part of the  $C_M$  surface lying inside  $C_0$  the electromagnetic potentials have finite values (as bremsstrahlung has reached these space regions).

In the negative  $z$  semi-space the experimentalist will detect only the *BS* shock wave. In the positive  $z$  semi-space, for the sufficiently large times ( $t > 2\alpha/(1 - \alpha^2)$ ) the observer close to the  $z$  axis will detect the Cherenkov shock wave  $C_M$  first, the *BS* shock wave  $C_0$  later and, finally, the shock wave  $C_L$  originating from the surpassing the medium light velocity. For the observer more remotod from the  $z$  axis the *BS* shock wave  $C_0$  arrives first, then  $C_M$  and finally  $C_L$  (Fig. 1). For large distances from the  $z$  axis the observer will see only the *BS* shock wave.

The positions of the shock waves for different observation times are shown in Fig. 2. The dimension of the Cherenkov cone is zero for  $t \leq t_l$  and continuously rises with time for  $t > t_l$ . The physical reason for this is that the  $C_L$  shock wave closing the Cherenkov cone propagates with the light velocity  $c_n$ , while the head part of the Cherenkov cone  $C_M$  attached to the charged particle

propagates with the velocity  $v > c_n$ . It is seen that for small observation times ( $t = 2$  and  $t = 4$ ) the  $BS$  shock wave  $C_0$  (pointed curve) precedes  $C_M$ . Later,  $C_M$  overtakes (this happens at the moment  $t = 2\alpha/(1 - \alpha^2)$ ) and partly surpasses  $BS$  shock wave  $C_0$  ( $t = 8$ ). However, the  $C_L$  shock wave is always behind  $C_0$  (as both of them propagate with the velocity  $c_n$ , but  $C_L$  is born at the later moment  $t = t_l$ ). The picture similar to the  $t = 8$  case remains essentially the same for later times.

### 3.2 Decelerated motion

Now we turn to the second problem (uniform deceleration of the charged particle along the positive  $z$  semi-axis up to a moment  $t = 0$  after which it rests at the origin). In this case, only negative retarded times  $t_i$  have a physical meaning (positive  $t_i$  correspond to the charge resting at the origin).

For the observation time  $t > 0$  the resulting configuration of the shock waves is shown in Fig. 3 where one sees the  $BS$  shock wave  $C_0$  arising from the termination of the charge motion (at the moment  $t = 0$ ) and the blunt shock wave  $C_M$  into which the  $CSW$  transforms after the termination of the motion. The head part of  $C_M$  is described with good accuracy by the sphere  $\rho^2 + (z - z_l)^2 = (t + t_l)^2$  corresponding to the fictitious shock wave  $C$  emitted from the point  $z_l = (1 - \alpha^2)^{-1/2} - 1$  at the moment  $t_l = -\alpha(1 - \alpha^2)^{-1/2}$  when the velocity of the decelerated charge coincides with the medium light velocity. Only part of  $C$  coinciding with  $C_M$  has physical meaning. The electromagnetic potentials vanish outside  $C_M$  (as no solutions exist there) and acquire infinite values on the internal part of  $C_M$ . Therefore, the surface  $C_M$  represents the shock wave. As a result, for the decelerated motion after termination of the particle motion ( $t > 0$ ) one has the shock wave  $C_M$  detached from a moving charge and the  $BS$  shock wave  $C_0$  arising from the termination of the particle motion. After the  $C_0$  wave reaches the observer, he will see the electrostatic field of a charge at rest and bremsstrahlung from remoted parts of charge trajectory.

The positions of shock waves for different times are shown in Fig. 4 where one sees how the acute  $CSW$ , attached to the moving charge ( $t = -2$ ), transforms into the blunt shock wave detached from it ( $t = 8$ ). Pointed curves

mean the *BS* shock waves described by the equation  $r = \alpha t$ .

For the decelerated motion and  $t < 0$  (i.e., before termination of the charge motion) physical solutions exist only inside the Cherenkov cone  $C_M$  ( $t = -2$  on Fig. 4). On the internal boundary of the Cherenkov cone the electromagnetic potentials acquire infinite values. On their external boundaries the electromagnetic potentials are zero (as no solutions exist there). When the charge velocity coincides with  $c_n$  the *CSW* leaves the charge and transforms into the  $C_M$  shock wave which propagates with the velocity  $c_n$  ( $t = 2, 4$  and  $8$  on Fig. 4). As it has been mentioned, the blunt head parts of these waves are approximated with a good accuracy by the surface  $\rho^2 + (z - z_l)^2 = (t + t_l)^2$  corresponding to the fictitious shock wave emitted at the moment when the charge velocity coincides with the light velocity in the medium.

In the negative  $z$  semi-space the experimentalist will detect the blunt shock wave first and *BS* shock wave (shock dash curve) later.

In the positive  $z$  semi-space, for the observation point close to the  $z$  axis the observer will see the *CSW* first and *BS* shock wave later. For larger distances from the  $z$  axis he will see at first the blunt shock  $C_M$  into which the *CSW* degenerates after the termination of the charge motion and the *BS* shock wave later (Fig. 4).

It should be mentioned about the continuous radiation which reaches the observer between the arrival of the above shock waves and about the continuous radiation and the electrostatic field of a charge at rest after the arrival of the last shock wave. They are easily restored from the complete exposition presented in [10] for the  $z = at^2$  motion law.

## 4 Conclusion

We have investigated the space-time distribution of the electromagnetic field arising from the accelerated manifestly relativistic charge motion. This motion is maintained by the constant electric field. Probably, this field is easier to create in gases (than in solids in which the screening effects are essential) where the Vavilov-Cherenkov effect is also observed. We have confirmed the intuitive predictions made by Tyapkin ([2]) and Zrelov et al. ([3]) concerning the exis-

tence of the new shock wave (in addition to the Cherenkov and bremsstrahlung shock waves) arising when the charge velocity coincides with the light velocity in medium. For the accelerated motion, this shock wave forms indivisible unity with Cherenkov's shock wave. It closes the Vavilov-Cherenkov radiation cone and propagates with the light velocity in the medium. For the decelerated motion, the above shock wave detaches from a moving charge when its velocity coincides with the light velocity in medium.

The quantitative conclusions made in [10] for a less realistic external electric field maintaining the accelerated charge motion are also confirmed. We have specified under what conditions and in which space-time regions the above-mentioned new shock waves do exist. It would be interesting to observe these shock waves experimentally.

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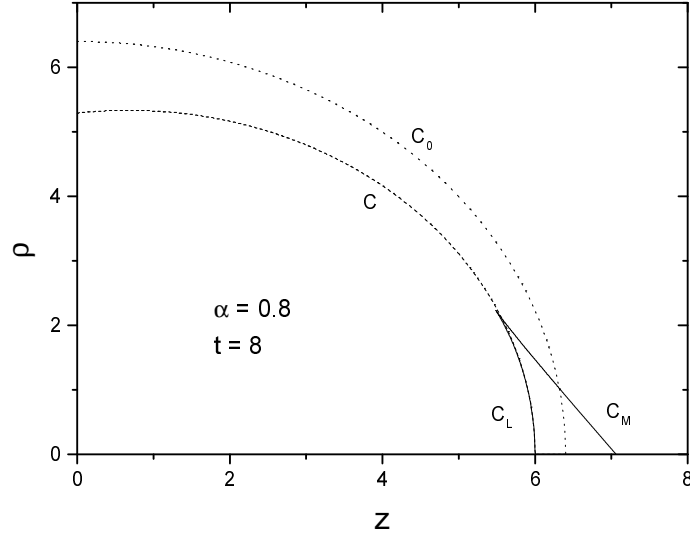


Figure 1: Typical distribution of the shock waves emitted by the accelerated charge.  $C_M$  is the Cherenkov shock wave,  $C_L$  is the shock wave emitted from the point  $z_l = (1 - \alpha^2)^{-1/2}$  at the moment  $t_l = \alpha(1 - \alpha^2)^{-1/2}$  when the charge velocity coincides with the medium light velocity. Part of it is described with good accuracy by the fictious spherical surface  $C$  ( $\rho^2 + (z - z_l)^2 = (t - t_l)^2$ ).  $C_0$  is the bremsstrahlung shock wave originating from the beginning ( at the moment  $t = 0$ ) of the charge motion.

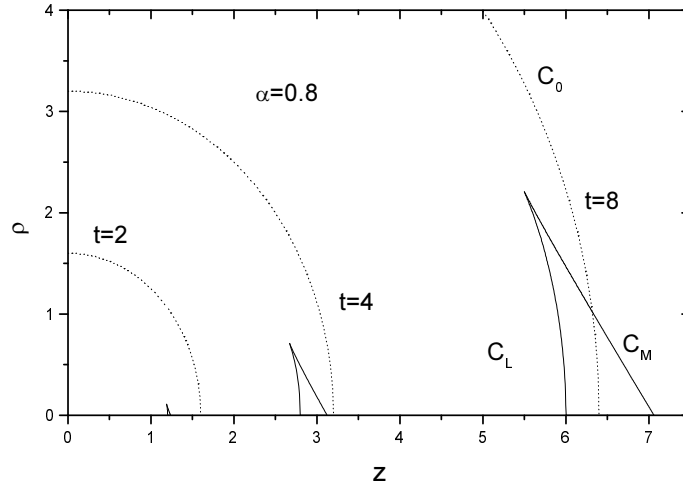


Figure 2: Time evolution of shock waves emitted by the accelerated charge.  $C_M$  and  $C_L$  are respectively the usual Cherenkov shock wave and the shock wave arising at the moment when the charge velocity coincides with the medium light velocity. Pointed curves are bremsstrahlung shock waves.

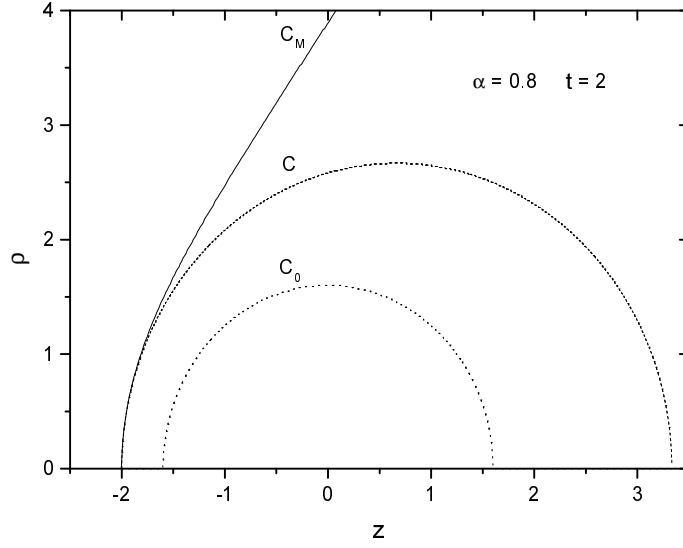


Figure 3: Space distribution of the shock waves produced by the decelerated charge in the uniform electric field.  $C_M$  is the blunt shock wave into which the  $CSW$  transforms after the moment when the charge velocity coincides with the medium light velocity. Part of it is approximated with good accuracy by the fictitious spherical surface  $C$ .  $C_0$  is the bremsstrahlung shock wave originating from the termination of the charge motion at the moment  $t = 0$ .

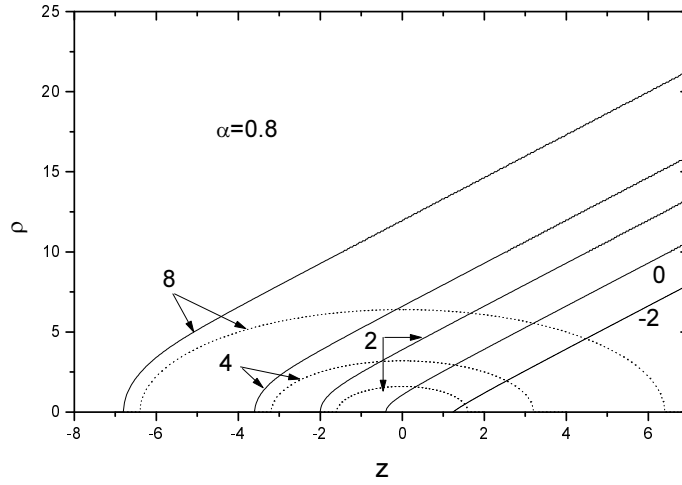


Figure 4: The continuous transformation of the acute Cherenkov shock wave attached to a moving charge ( $t = -2$ ) into the blunt shock wave detached from a charge ( $t = 8$ ) for the decelerated motion. The numbers at the curves mean the observation times. Pointed curves are bremsstrahlung shock waves. Charge motion is terminated at the moment  $t = 0$ .